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Hypercomplex Number Approach to Schwinger's Quantum Source Theory

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Abstract

The hypercomplex numbers associated with the Dirac–Clifford algebra, are applied to the spin-0, -$\frac{1}{2}$, and -1 structures for Schwinger's source theory of quantum electrodynamics. The generalizations to 5-vectors and 5-space relativity are introduced. The anticommuting numbers, associated with Fermi–Dirac statistics, are examined in some detail. The hypercomplex number formulation is suitable for curved space quantum computations.

1. *Introduction*

In our recent explorations of the hypercomplex number structure of relativistic quantum physics (Edmonds, 1974, 1975), we considered only coupled partial differential equations. Though this is very important, it is only the first step in developing a quantum theory. This is because the fields, described by the partial differential equations, are *not observable* by humans. Only in nonrelativistic quantum physics (chemistry/physistry) is it useful to consider $|\psi|^2$ as the probability distribution for "finding" the electrons. This leads to an atomic orbital picture with bumps (and directions) that are quite helpful in understanding molecular structure. The hydrogen atom may not, however, be a physical proton, of mass M_p, coupled to an orbiting physical electron, of mass M_e, by an electromagnetic photon field, of mass $M_\gamma = 0$. A hydrogen atom is a stable, spin-0 "quantum," of mass $M_H < M_p + M_e$. This "explains" why it does not decay into $e^- + p^+$, in the same way that $M_p < M_n + M_e$ "explains" why $p^+ \to n + e^+$ does not occur. It is well known that $n \to p^+ + e^- + \bar{\nu}$, yet we certainly do not "think" of the neutron as composed of orbiting $p^+, e^-, \bar{\nu}$ constituents. One can argue elegantly that it is no more "correct" to think of hydrogen in terms of a few simple parts in orbit.

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But once we admit that hydrogen is a quantum, then H_2 is also. There is no stopping; even the earth is a quantum! Clearly this is not a useful concept. We must find some line between quanta and particles. There is a useful concept for drawing this line. We know that protons have a radius of 1.2 fermis, in the sense that they pack in nuclei as if this were their size. This is the only real measure of the size of quanta. We know dimensionally that $\hbar/mc$ has the units of length. We, therefore, postulate that $\lambda m \equiv \text{const} = 2(1.2) \times 10^{-15}\text{m} \times 1.7 \times 10^{-27}$ kg, holds for all quanta. This gives the peculiar result that the heavier a quantum is, the smaller it is. Thus protons and neutrons are $\lambda = 2.4$ F in diameter and electrons are $(2.4 \text{ F}) (M_p/M_e) \approx 0.48 \times 10^{-2}$ Å $\approx 10^{-2} R_0$, the Bohr radius. We now define a *particle* to be a massive system that has an experimental size much larger than the quanta with the highest amplitude for being its constituents. Thus, $H \to e^- + p^+ \to H$, virtually, is most likely and experimentally H is about 1 Å in diameter, when packed into a solid form. This gives a volume about $(10^2)^3 = 10^6$ times that of the $e^- p^+$ combination. Thus, by our definition, hydrogen and all atoms are composite particles. Notice that $n \to pe\nu \to n$, virtually, has $\lambda_n < \lambda_e$, therefore n is a quantum (consider also $n \to p^+\pi^- \to n$).

When particles are broken, their parts rearrange themselves, e.g., $H_2O \to H + H + O$, whereas when quanta decay, the products are *created*, e.g., $n \to p^+ + e^- + \bar{\nu}$. Particles, or clusters of particles, are directly observable for humans and should be considered as the sources and sinks for quanta. This is not aesthetically satisfying, since we would like to think of nature having structure and interaction only at the quantum level. But physics is not yet ready to attack nature in this way.

The point is that physics has had to face the mathematically difficult idea of particle creation and destruction, something which coupled partial differential equations cannot capture. Perhaps at some deeper level of dynamics the "partons" are indestructable and all "observed" quanta can be described in analogy to the way we build up the periodic chart. This is just not known at present. Quantum electrodynamics is successful to six or eight digits and assumes that quanta are actually created by sources and annihilated by sinks. A *quantum* is a set of quantum numbers moving through space-time in an approximately localized fashion. It can "be," momentarily, any combination of other quanta that together have the specified net values of the quantum numbers. From this point of view, the hydrogen atom is a virtual swarm of *all* the quantum "particles" known to physics. (The boot-strap philosophy carries this a step farther and takes as an article of faith the idea that "self-consistency" dictates the observed mass spectrum and quantum numbers of the "observed" quanta, whatever that means.) The hydrogen atom has a much larger probability *amplitude* to "exist" as a (p^+e^-) state than as a $(n\pi^+e^-)$ or $(p^+e^-e^+e^-)$, etc., state, but these other possibilities do contribute to its behavior, when measurements are made to several digit accuracy.

The high-precision measurements are always classical many-body procedures. Humans cannot perceive anything else. No quanta are ever "seen." The "recoil" of a macro-body (particle or collection of particles) is observed, and *classical* relativistic mechanics is then used to assign quantum numbers to the quantum responsible for the recoil. We say that a quantum was thus created (prepared) or annihilated (measured).

Schwinger (1970) has beautifully developed this pragmatic, macro-oriented approach to quantum electrodynamics. He bypasses second quantization and and renormalization, while obtaining all the predictive power of tranditional quantum electrodynamics. The approach has special appeal for me, because it seems to be most naturally compatible with the hypercomplex number approach and because it avoids assumptions about microscopic regions of space-time, where I suspect curvature and higher dimensions will play an important role in future parton developments.

Schwinger's approach builds on the mathematical form

$$\langle 0_+ | 0_- \rangle^S = \exp \frac{i}{2} \int \text{field}(x)\text{source}(x)(dx) \tag{1.1}$$

where "field (x)source (x)" is a function of space-time, assumed invariant under Lorentz symmetry $[(x) \leftrightarrow \{x^\mu\}]$, and $(dx) \equiv dx^0 dx^1 dx^2 dx^3$ describes integration over space and time. The complex number $\langle 0_+ | 0_- \rangle$ is identified with the probability amplitude for the creation, propagation, interaction, and destruction of a particular set of quanta, prepared and destroyed by measured "sources." The specification of "source(x)" is where the dynamics enters.

We shall not pursue this important, and still developing, aspect of the theory. In this paper we shall explore the possible structures for the term "field(x)source(x)" that are suggested by the hypercomplex number approach to the field equations. This formulation will also generalize in a natural way to integrating over *curved* space-time, but we shall not discuss that here. We shall, near the end, consider the anticommuting numbers that Schwinger uses, in the integral appearing in equation (1.1), to build in Fermi-Dirac statistics.

2. *The Hypercomplex Operator*

We assume that the hypercomplex number system $\{e_\mu, ie_\mu, f_\mu, if_\mu\}$ with complex coefficients, is central to the theory. We further assume that

$$P \equiv i\hbar(e_\mu)\partial^\mu + i\hbar(if_0)\partial^4 \equiv P^\dagger \to L^\dagger P L \tag{2.1}$$

is the central differential operator of the theory. A ten-parameter generalization of Lorentz symmetry is assumed, to restrict allowed wave equations, in the form $LL^\wedge \equiv (e_0)$. For simplicity we define

$$P \equiv P^a b_a, \qquad a = 0,1,2,3,4, \ \{b_a\} = \{e_\mu, (if_0)\} \tag{2.2}$$

It is easy to show that

$$(P|P) \equiv \tfrac{1}{2}[(P^\wedge P) + (\ )^\wedge] = P^a P_a = P^\mu P_\mu + P^4 P_4 \to L^\wedge (P|P) L = (P|P) \tag{2.3}$$

$$\begin{aligned} (b_a | b_c) &= +1, & a = c = 0 \\ &= -1, & a = c \neq 0 \\ &= \ \ 0, & a \neq c \end{aligned}$$

Though we shall not really need to consider the detailed structure of the operator P, a comment is in order on the unusual term $i\hbar(if_0)\partial^4$. At our present primitive level of understanding of quantum physics, we do not know how to deal with this term. Instead we must replace it by $mc(if_0)$, where m is the empirical mass quantum number of the quanta created by the sources we can manipulate. The macro-sources have measurable (classical) mass and are humanly observed to move in three-dimensional space as a "unit." Emissions and absorptions of quanta cause abrupt changes in these classical motions. We use these changes to *assign* a mass number to the quanta involved, though the quanta are *never* directly seen. We cannot be sure that the quanta themselves do not propagate in five-dimensional space-time. The structure of the natural hypercomplex number system suggests five-space rotations as the natural group symmetry, with the (if_0) coordinate different from the other four (e_μ) in some fundamental way. It probably contains the scale setting length in nature.

3. *The Spin-$\frac{1}{2}$ Field*

The simplest covariant wave equation has the form

$$P\psi = \eta(x), \qquad \psi \to L^{\wedge}\psi, \qquad \eta \to L^{\dagger}\eta \tag{3.1}$$

For historical reasons, this is called spin-$\frac{1}{2}$. The field $\psi(x)$ and its source $\eta(x)$ are both multiplied *only* from the left, and thus have a 1 x 4 matrix representation. We can express this as

$$\psi \equiv \psi^a g_a, \qquad \eta \equiv \eta^a g_a, \qquad a = 1,2,3,4 \tag{3.2}$$

An explicit representation for $\{g_a\}$ in terms of $\{e_\mu, ie_\mu, f_\mu, if_\mu\}$ can be obtained by noting that $i(ie_3)(if_0) = (if_0)i(ie_3)$. Thus ψ can be chosen as an eigenstate of these when $i\hbar\partial^k\psi = 0$, $k = 1,2,3$. These are called spin up/down and particle/antiparticle states. Once $\{g_a\}$ is chosen, we can easily work out the multiplication table for this set and $\{e_\mu, ie_\mu, f_\mu, if_\mu\}$. For those wave functions that have $i\hbar\partial^4\psi = 0$, we choose $i(ie_3)$ and $i(ie_0)$ to obtain eigenstates.

Now we define the corresponding integral equation for ψ:

$$\psi(x) \equiv \int G(x,x')\eta(x')\,(dx'), \qquad (dx') \equiv dx^{0'} \cdots dx^{4'} \tag{3.3}$$

For L symmetry covariance, we must have

$$G \to L^{\wedge}GL^{\dagger\wedge} \Rightarrow G^{\dagger} \equiv \pm G \text{ is possible} \tag{3.4}$$

Therefore, we can define

$$G \equiv G^{ab}g_a g_b^{\dagger}, \qquad G^{ab} \equiv \pm G^{ba}$$

To obtain the equation satisfied by G, we operate on equation (3.3)

$$P(x)\psi(x) = \int P(x)G(x,x')\eta(x')\,(dx') = \eta(x) \tag{3.5}$$

which is consistent with

$$P(x)G(x,x') \equiv \delta(x,x')\,(e_0) \tag{3.6}$$

Since the right-hand side is proportional to (e_0) and $PP^{\hat{}}$ is also proportional to (e_0) we are led to define

$$G \equiv P^{\hat{}}\Delta(x,x') \to L^{\hat{}}P^{\hat{}}L^{\dagger\hat{}}\Delta(x,x')' \equiv L^{\hat{}}P^{\hat{}}\Delta(x,x')L^{\dagger\hat{}} \tag{3.7}$$

Therefore, $\Delta(x,x')$ is invariant and proportional to (e_0). It satisfies

$$PP^{\hat{}}\Delta(x,x') = \delta(x,x')\,(e_0) = P^a P_a(e_0)\Delta, \qquad a = 0,1,2,3,4 \tag{3.8}$$

Now we define $\delta(x,x')^\dagger \equiv \delta(x,x')$, which leads to $\Delta = \Delta^\dagger$ and

$$(PG)^\dagger = (PP^{\hat{}}\Delta)^\dagger = G^\dagger \overleftarrow{P} = \Delta P^{\hat{}} \overleftarrow{P} = \delta(x,x')\,(e_0) \tag{3.9}$$

So, for consistency, we must have

$$G^\dagger \equiv +G, \qquad G^{ab} \equiv +G^{ba}, \text{ or alternatively } G \equiv G^a b_a^{\hat{}} \tag{3.10}$$

Now we are prepared to guess at the proper form for field (x) source $(x) \propto (e_0)$ and invariant, in equation (1.1): We try

$$\int \text{field}(x)\,\text{source}(x)\,(dx) = \int \psi(x)^\dagger \eta(x)(dx) \tag{3.11}$$

This works because $\psi(x)^\dagger\eta(x) \to (L^{\hat{}}\psi)^\dagger L^\dagger \eta = \psi^\dagger L^{\hat{}\dagger} L^\dagger \eta = \psi^\dagger \eta$, and $(\;)^\dagger$ is like Hermitian conjugation, and ψ, η have 1 x 4 matrix representations. As stated earlier, the dynamics enters through the source terms. The Green's function G can be obtained from $\Delta(x,x')$ and appropriate boundary conditions. We, therefore, express the transition amplitude totally in terms of the unspecified sources:

$$\int \psi^\dagger \eta = \int \eta(x')^\dagger G(x,x')\eta(x)\,(dx')(dx) \tag{3.12}$$

where $G \equiv G^\dagger$ has again been used. One further assumption is necessary. What number system do we use for dx, G^{ab}, and η^a? The ordinary real and complex numbers are used for dx and G^{ab}, but, with Schwinger, we make the drastic assumption that nature requires *another* system for $\eta^a(x)$. This "complex" system has anticommuting numbers, $ab = -ba$, $aa = -aa = 0$. (!) This is how Schwinger introduces Fermi–Dirac statistics for the quanta described by $\psi(x)$. Notice that a pair of Schwinger numbers, ab, commutes with all other numbers and, therefore, $\int \psi^\dagger \eta$ is an ordinary number, though ψ and η are not. We also define the Schwinger numbers to commute with the hypercomplex numbers:

$$\eta(x) \equiv \eta^a(x) g_a \equiv g_a \eta^a(x) \tag{3.13}$$

The expression $\eta^a(x)$ represents a mapping from the real numbers $(x^0, x^1, x^2, x^3, x^4)$ to the Schwinger numbers $(\eta^1, \eta^2, \eta^3, \eta^4)$. We shall consider these further at the end of our discussion.

4. *The Spin-*0 *Field*

The next simplest covariant wave equation has the form

$$PP^{\hat{}}\phi = K(x), \qquad \phi \to \phi, \qquad K \to K, \qquad PP^{\hat{}} \to PP^{\hat{}} \tag{4.1}$$

For historical reasons, this is called spin 0. The field ϕ and its source K can be thought of as "representations" of the form

$$\phi \to \phi' \equiv L^{\dagger}\phi L^{\dagger\wedge} \equiv L^{\dagger}L^{\dagger\wedge}\phi = \phi \Rightarrow \phi \propto (e_0) \tag{4.2}$$

The question now arises as to whether ϕ should be an eigenstate of some particle-antiparticle operator, such as $i(ie_0)$. This does not appear to fit the properties of ϕ above, so probably this field should represent spin-0 particles which are their *own* antiparticle, such as π^0.

It is easy to see that the integral equation for $\phi(x)$ is given by

$$\phi(x) \equiv \int \Delta(x, x')K(x')(dx') \tag{4.3}$$

since this gives

$$PP^{\wedge}\phi(x) = \int (PP^{\wedge}\Delta)K(dx') = \int \delta(x, x')K(x')(dx') \tag{4.4}$$

using equation (3.8). It is generally considered that spin 0 is the simplest field to describe. However, $PP^{\wedge}$ is *not* simple in *curved* space. The proper covariant derivative must be defined so that $P^{\wedge} = \mathscr{D}^a b_a(x)^{\wedge}$ is operated on correctly by P.

Finally then, we guess at the transition amplitude expression

$$\int \text{field}(x)\text{source}(x) \equiv \int \phi(x)K(x)(dx) = \int K(x')\Delta(x, x')K(x)(dx')(dx) \tag{4.5}$$

Is this all that can be said about spin 0? What about the quanta $\pi^+\pi^-$, where there are distinct antiparticles? Above we made the strong assumption that $\phi \equiv \phi^{\dagger} \equiv \phi^{\wedge}$. We should also consider the more general possibility $\phi \equiv \phi^{\wedge}$, which means that ϕ is proportional to $\{e_0, ie_0, f_\mu\}$. But (e_0) is invariant, so we have the 5-vector field $\tilde{\phi} \to L^{\dagger}\tilde{\phi}L^{\vee\dagger}$, $\tilde{\phi} \equiv +\tilde{\phi}^{\wedge}$. That ϕ is a 5-vector follows from $(f_0) \times \{ie_0, f_\mu\} = \{if_0, e_\mu\}, f_0L^{\dagger} = L^{\wedge}f_0$, and the fact that $(\{e_\mu, if_0\}$ is the basis for the 5-vector $P \to L^{\dagger}PL)$. Since $L^{\dagger}(ie_0)L^{\vee\dagger} = L^{\dagger}(L^{\wedge\vee})^{\vee\dagger}(ie_0) = L^{\dagger}L^{\vee\dagger}(ie_0)$, we see that if $L^{\vee} = L^{\wedge}$, then (ie_0) would be invariant. This is the six-parameter Lorentz subgroup, $LL^{\wedge} = LL^{\vee} = (e_0)$. Traditionally, this has been assumed to be the physical group. In this case (e_0) and (ie_0) are both invariant. For this restricted group

$$\phi \equiv \phi_+[e_0 + i(ie_0)] + \phi_-[e_0 - i(ie_0)] \tag{4.6}$$

would give ± 1 eigenstates of $i(ie_0)$, corresponding to particles and antiparticles.

This could be taken as evidence that the restriction $L^{\wedge} = L^{\vee}$ is physically necessary, since charged, spin-0 pions exist! However, it was *not* necessary to assume that $\phi \to L^{\dagger}\phi L^{\dagger\wedge}$. Only the left-hand $L^{\dagger}$ is required to match the $PP^{\wedge}$ transformation. The possibility $\phi \to L^{\dagger}\phi$ will not be considered, since this is essentially spin $\frac{1}{2}$. We should consider, though,

$$\theta \to \theta' \equiv L^{\dagger}\theta L \equiv \theta L^{\wedge}L = \theta \equiv +\theta^{\dagger} \propto (f_0) \tag{4.7}$$

which also gives an invariant representation for the full 10-parameter group. It is possible, therefore, to postulate that the $\pi^+\pi^-$ quanta are mixtures of the ϕ and θ fields. In this case, the particle-antiparticle operator is (f_0), since

$$(f_0)[e_0 + f_0] = +1[e_0 + f_0] \text{ and } (f_0)[e_0 - f_0] = -1[e_0 - f_0] \tag{4.8}$$

We therefore guess that the spin-0 transition amplitude also contains

$$\int S(x')\Delta(x,x')S(x)(dx')(dx), \qquad S(x) \to L^{\dagger}SL \equiv L^{\dagger}L^{\dagger\hat{}}S \equiv S \propto (f_0) \quad (4.9)$$

with

$$\theta(x) \equiv \int \Delta(x,x')S(x')(dx'), \qquad S(x') \propto (f_0) \qquad (4.10)$$

For Bose-Einstein statistics, we assume that K and S have ordinary, commuting number coefficients. Notice that we have two distinct types of spin-0 quanta, corresponding to e^-e^+ (spins opposite), and e^-e^- (spins opposite).

5. *The Spin*-1 *Field*

Just as we have formulated two covariant solutions to the spin-0 wave equation, by altering the covariance structure of the source, we can replace $\eta \to L^{\dagger}\eta$ by $J \to L^{\dagger}JL$, to obtain another solution to the Dirac equation

$$PF \equiv J, \qquad J \to J' \equiv L^{\dagger}JL, \qquad \Rightarrow F \to F' = L^{\hat{}}FL \qquad (5.1)$$

We have two special forms, $F = \pm F^{\hat{}}$. For $F = +F^{\hat{}}$, we have the possibility $F \propto e_0$ and invariant, and the possibility of F containing $\{ie_0, f_\mu\}$, which gives a 5-vector. The other possibility, $F = -F^{\hat{}}$, contains $\{e_k, ie_k, if_\mu\}$. Traditionally, the e_k and ie_k components are associated with E^k and B^k:

$$E^k(e_k) + B^k(ie_k) \equiv \tfrac{1}{2}(F + F^{\check{}\hat{}}), \qquad F \equiv -F^{\hat{}} \qquad (5.2)$$

Under the Lorentz subgroup, (ie_0) is invariant and $\{e_k, ie_k\}$ will mix with itself as does $\{f_\mu\}$.

We now split the source into two parts, $J \equiv J_c + J_N$, and compute

$$P^{\hat{}}PF = P^{\hat{}}J = (P|P)F \Rightarrow (P|P)(F + F^{\hat{}}) = (P|J) = (P|J_c) + (P|J_N) \qquad (5.3)$$

We conclude that $F \equiv -F^{\hat{}}$ has a "conserved" source $(P|J_c) \equiv 0$. We shall concentrate only on this type of source, since it apparently applies to electromagnetism. Actually, to this point we have only required that $J \to L^{\dagger}JL$, which gives the simple possibilities $J = \pm J^{\dagger}$. Electromagnetism further assumes that $J \equiv +J^{\dagger}$ is appropriate, as a source restriction, and $(P|J) \equiv 0$.

Because $F = -F^{\hat{}}$ and $F \to L^{\hat{}}FL^{\dagger\hat{}}$, we could write F in the form

$$F \equiv F^{ac}b_a^{\hat{}}b_c, \qquad F^{ac} \equiv -F^{ca} \qquad (5.4)$$

since $b_a \to L^{\dagger}b_aL$ and $(b_a^{\hat{}}b_c)^{\hat{}} = b_c^{\hat{}}b_a$.

Next we try to construct an integral equation for F, with source $J \equiv J^ab_a \equiv J^{\dagger}$ and $(P|J) \equiv 0$. The second condition is met by the form

$$F \equiv \tfrac{1}{2}\int [H(x,x')J(x') - J^{\hat{}}H^{\hat{}}]\,(dx') \Rightarrow H \to H' \equiv L^{\hat{}}HL^{\hat{}\dagger} \qquad (5.5)$$

Notice next that H transforms as $P^{\hat{}}$. We could try $H \equiv H^{ab}g_ag_b^{\dagger}$, $H^{ab} \equiv +H^{ba}$, but by analogy with the spin-$\frac{1}{2}$ case we are tempted to try

$$H(x,x') \equiv P(x)^{\hat{}}\Delta\!\!\!\!\!\!\;|\,(x,x'), \qquad \Delta\!\!\!\!\!\!\;|\,(x,x') \propto (e_0) \qquad (5.6)$$

Then $H = H^{\dagger}$, and we must find the equation satisfied by $\Delta(x, x')$. This must come from the field equation

$$PF = J = \int J(x')\delta(x, x')(dx') = \tfrac{1}{2}\int [(PP^{\hat{}}\Delta)J - PJ^{\hat{}}P\Delta]\,(dx') \qquad (5.7)$$

Here $J = J^a(x')b_a$ is an independent function except that $P^a(x')J(x')_a = 0$. Therefore, we should be able to extract a second-order equation for $\Delta(x, x')$ from equation (5.7). Notice that $P = P(x)$ and $J = J(x')$, so P does *not* operate on J in equation (5.7). We can use $b_a^{\hat{}}b_c = 2(b_a | b_c) - b_c^{\hat{}}b_a$ on the second term $PJ^{\hat{}}P$, if useful.

It is easy to check, by direct computation, that HJ in the form $\{e_\mu, if_0\} \times \{e_\mu, if_0\}$ gives $\{e_0, e_k, ie_k, if_\mu\}$. Thus the troublesome second term in equation (5.5) is needed to remove only the (e_0) terms in HJ.

Now we are ready to guess at the invariant probability amplitude expression field(x) source(x). The simplest possibility would be

$$\int J(x')^a \Delta(x, x') J(x)_a (dx')(dx) \qquad (5.8)$$

This is both invariant and proportional to (e_0). However, it is not "elegant" in the sense that the full hypercomplex numbers do not appear. They are probably necessary in *curved* space, so we would like to have an expression in terms of J and H. A choice like

$$\tfrac{1}{2}\int [J(x')^{\hat{}}H(x, x')^{\hat{}}f_0 J(x)] + [\quad]^{\hat{}} \qquad (5.9)$$

would be in the spirit of equation (5.8), except that $L^{\hat{}}(A + A^{\hat{}})L$ is equal to $A + A^{\hat{}}$ only if $A + A^{\hat{}}$ is proportional to (e_0). For quaternions, there is no problem. However, rest mass required us to extend the quaternions from 8- to a 16-part number system. Now $(A + A^{\hat{}}) = (\quad)^{\hat{}}$ implies, in general, only that $A + A^{\hat{}} \leftrightarrow \{e_0, ie_0, f_\mu\}$. We can remove the (f_μ) possibility by the form $(A + A^{\hat{}}) + (\quad)^{\check{}}$ but this introduces $(L^{\hat{}}(\quad)L)^{\check{}} = L^{\check{}}(\quad)L^{\hat{}\check{}}$. Again, we see that this approach is consistent only for the Lorentz subgroup, where $L^{\hat{}} \equiv L^{\check{}}$. Since $\langle 0_+ | 0_- \rangle$ is directly related to predictions about macro-measurements made by humans, who only see three-dimensional space, perhaps it is appropriate to require that $\langle 0_+ | 0_- \rangle$ be only Lorentz invariant, whereas we postulate that the field equations are $LL^{\hat{}} = (e_0)$ covariant. Note, however, that $Ff_0 J \to \{e_k, ie_k, f_\mu\} \times f_0 \times \{e_\mu, if_0\} = \{e_0, ie_k, e_k, if_\mu\}$, so the form $\tfrac{1}{2}\int [(Ff_0 J) + (\quad)^{\hat{}}]$ is preferable to equation (5.9).

6. *The Spin-1′ Field*

The spin-1 field equation, $PF = J$, contains $i\hbar\partial^4(if_0)$ and can, therefore, have particle-antiparticle eigenstates of (if_0). [If the quanta are massless, $i\hbar\partial^4 F \equiv 0$, then (ie_0) commutes with P and can be used for the particle-antiparticle operator.] Such quanta are analogous to two spin-up electrons or two anti-electrons. As with spin-0, we also have the kind of quanta anloagous to an electron and position, both spin up. This quantum is its own antiparticle, and hence has less "internal" structure. We might, therefore, expect a second-order wave equation. But $PP^{\hat{}}\phi = K$ has already been used for spin-0. The other spin-1 field F has a source $J \to L^{\dagger}JL$. If we adopt this for our new spin-1 field, A, we should start with

$$PP\hat{}A = J = P\hat{}PA = (P|P)A \tag{6.1}$$

This requires that $A \to L^{\dagger}AL$. Again we have the choice $J = \pm J^{\dagger}$. This would result in $A = \pm A^{\dagger}$, which requires $\{e_\mu, if_0, f_0\}$ or $\{ie_\mu, f_k, if_k\}$. Since F already has 10 components and A should be "simpler," we choose $A \equiv +A^{\dagger}$, which means that $J = +J^{\dagger}$. Again we reject $J \propto (f_0)$, since this is invariant under $L^{\dagger}(\quad)L$ and represents spin-0. We therefore assume that J and A are 5-vectors. But then $(P|A) \propto (e_0)$ is invariant, so that $PP\hat{}A$ could be supplemented by $P(P|A)$. Again a choice must be made.

The spin-1 field F had $(P|J) \equiv 0$ in order that $F = -F\hat{}$. We shall *assume* that $(P|J) = 0$ holds for the field A also. This requires

$$(P|P)A - P(P|A) \equiv J = A(\underset{\leftarrow}{P|P}) - P(P|A) \tag{6.2}$$

In electrodynamics, one usually chooses instead

$$(P|P)A \equiv J, \qquad P(P|A) \equiv 0 \tag{6.3}$$

and calls the second equation the Lorentz gauge condition. Equation (6.2) is a weaker condition on A and more compatible with the form of the other wave equations we have constructed. I suspect that it is more correct in the fully coupled (curved space) field theory.

Now we are ready to construct an integral equation for A. We try

$$A(x) \equiv \tfrac{1}{2}\int [J(x')D(x,x') + (JD)^{\dagger}]\,(dx') \Rightarrow D \to L\hat{}DL \tag{6.4}$$

Now $D = \pm D\hat{}$ is a possibility. Further, for $D = +D\hat{}$ we can have $D \to L\hat{}DL \equiv L\hat{}LD$, which would mean D proportional to (e_0) and invariant. This is traditionally done, but it requires that A be "parallel" to J, which seems reasonable from equation (6.3), but not from equation (6.2). The general case $D = +D\hat{}$ would produce $\{e_\mu, ie_\mu, f_\mu, if_\mu\}$ in the product JD and the term $(JD)^{\dagger}$ would leave $\{e_\mu, if_0, f_0\}$. The term (f_0) would make $(P|A)$ not proportional to (e_0). This problem does not arise with the choice $D \equiv -D\hat{}$ since $\{e_\mu, if_0\} \times \{e_k, ie_k, if_\mu\} = \{e_\mu, ie_\mu, if_\mu, f_k\}$. Only the term (f_0) is missing. Now adding $(JD)^{\dagger}$ would leave $\{e_\mu, if_0\}$, as desired for A. We, therefore, assume that $D \equiv -D\hat{}$.

By combining equation (6.2), equation (6.4), and $J = \int \delta(x,x')J(x')(dx')$, we can construct a second-order equation for $D \equiv \frac{1}{2}D^{ac}\hat{b}_a b_c$, $D^{ac} \equiv -D^{ca}$, along the lines used for $\triangle\!\!\!\!\triangle(x,x')$.

Now we consider the probability amplitude expression. We guess

$$\int (A|J)(dx) = \tfrac{1}{4}\int J(x')D(x,x')J(x)\hat{}(dx')(dx) + \cdots \tag{6.5}$$

for $\int$ field(x) source(x).

This completes our outline of the spin-1 system. The question naturally arises as to how F and A are related. Both have 5-vector sources J that are "conserved." Do they invariably have the same source, so that both quanta are always produced together? We could guess that

$$F = \tfrac{1}{2}[(P\hat{}A) - (\quad)\hat{}] = -[\quad]\hat{} \tag{6.6}$$

which would lead to (Edmonds, 1975)

$$PF = P\tfrac{1}{2}[(P\hat{A}) - (\quad)\hat{}\,] = J_A \tag{6.7}$$

This would make $J_A = J_F$, which should make one of the fields unnecessary. In fact, the traditional coupling $P\psi \to (P - \epsilon A)\psi$ suggests that F is unnecessary. But perhaps F is more general than equation (6.6) would suggest or perhaps J_A is a 6-vector, or is not conserved, so that equation (6.6) is not valid. There should be two types of spin-1 quanta, some that are their own antiparticle and some that are not. I, therefore, remain hopeful that some basic distinction will be found between F and A, e.g., $J_F \equiv J_A + J_N$, $J_N^\dagger \equiv -J_N \neq 0$, so that $F \neq -F\hat{}$, even though $(P|J_A) = 0$.

7. *Anticommuting Numbers*

The ordinary complex numbers are built on several assumptions: $a(b + c) \equiv ab + ac$, $a + b \equiv b + a$, $ab \equiv ba$, $0(a) \equiv 0$, $1(a) \equiv a$, $a + 0 \equiv a$, $a + (-a) \equiv 0$, $(a^{-1})(a) \equiv 1$, $ia \equiv ai$, $ii \equiv (-1)$, $(ab)^* \equiv b^*a^* = a^*b^*$, $(i)^* \equiv (-i)$, and $\exp(a) \equiv \Sigma a^n/n!$. This particular number system has proven to be a powerful aid for engineering and classical physics. It now appears that it is "insufficient" for quantum physics. To build up the complex numbers, we begin with $\{1, 0, -1, i, 0i, -i\}$. We can now *prove* that $(-)(-) = (+)$ and define $2 \equiv 1 + 1$, $-2 \equiv -1 + -1$, $2i \equiv (i + i)$, and $(-2i) \equiv (-i) + (-i)$. Actually, only $2 \equiv 1 + 1$ is necessary, since $i(2) = i(1 + 1) = i + i$, etc., follow from the distributive law. The important thing is that 2 is not equal to any of the other numbers already existing. Then $2 + 1$ is again a new quantity, and we generate a countably infinite set of numbers this way.

The rational numbers form a dense, closed algebra and naturally fit the concept of position in one dimension, $-\infty < x < +\infty$. The imaginary numbers can be motivated by the desire to find numbers that, times themselves, give all the real numbers. Thus $(i)(i) \equiv (-1)$ is sufficient to do this. If we then try to construct numbers that times themselves give the imaginary numbers, we find that they exist within the complex numbers, formed by a linear combination of real and imaginary numbers. But what about $(0i)$? Clearly $(0i)(0i) \equiv (0)(0)(i)(i) = (0)(-1) = 0$, and $(0)(0i) \equiv (0)(0)(i) = (0i)$. Now, we know that $(0)(5) = 0$ and $0(5i) = 0(5)(i) = 0(i) = (0i)$. Can we say that $(0i) \stackrel{?}{=} (0)$? If so, then the imaginary number line is not entirely distinct from the real number line, or alternatively the imaginary number line has no zero position on it. One can sense an incompleteness in the number system so defined. Let us define $(0i) \neq (0)$ so that the imaginary number line is *completely distinct* from the real number line. Once we do this, we find our system is still incomplete in the sense that no complex number exists, which times itself gives $(0i)$; $x^2 \equiv (0i)$. This is like the original problem, $x^2 = (-1)$, which required the introduction of the imaginaries. The natural solution is the introduction of another set of numbers, incredible as it may seem. In analogy with the way the imaginaries got their name, we shall call these the *incredible numbers.* We define $(\delta)(\delta) \equiv (0i)$ and $i(\delta) \equiv (i\delta)$. Then $(i\delta)(\delta) = i(\delta\delta) = i(0i) = ii(0) = 0$.

Therefore, $\delta \neq i\delta$. We have both incredibles, and imaginary incredibles, and an incredible, times an imaginary incredible, gives a real number. Since $(0)(i) \equiv (0i)$, we define $(0)(\delta) \equiv (0\delta)$ and $(\pm 1)(\delta) \equiv (\pm\delta)$. Therefore, $\delta + (-\delta) \equiv [1\delta + (-1)\delta] \equiv (0)\delta = (0\delta) \neq 0 \neq (0i) \neq (0i\delta)$. The question then arises as to whether there is any difference between δ and (0δ). Notice that $(0\delta) = (0)(\delta) = (-0)\delta = (-1)(0)\delta = (-1)(0\delta) = -(0\delta)$, whereas $\delta = (1\delta) = (1)\delta \neq (-1)\delta = (-\delta)$. So they *are* distinct. We conclude that $x^2 = (0i) = -(0i)$ has six solutions: $x = \delta, -\delta, (0\delta), (i\delta), (-i\delta)$ and $(0i\delta)$.

The next thing to do, would be to try and expand the incredible number system. We naturally define $2\delta \equiv (\delta + \delta), \delta(\delta + \delta) \equiv \delta^2 + \delta^2$. Then $(2\delta)(2\delta) = 4\delta\delta = 4(0i) = (0i)$. Also $(-\delta)(2\delta) = 2(-\delta)(\delta) = -2(\delta\delta) = -2(0i) = (0i)$. In every way 2δ acts like δ, so we can conclude that $2\delta = \delta$. Recall that $(N)(0) = 0$ but $(\infty)(0) = ?(0 \text{ or } \infty)$. Similarly, $N\delta = \delta$ but $(\infty)(\delta) = ?$ By now you have probably noticed a similarity between $\{\delta, 0\delta, -\delta, i\delta, 0i\delta, -i\delta\}$ and the physicists' infinitesimal $\{dx, 0dx, -dx, idx, 0idx, -idx\}$. The only difference is that $(dx) \times (dx) \approx 0$ instead of $\approx (0i)$, but this amounts to the same thing in all practical calculations. Mathematicians call the rigorous use of differentials "nonstandard mathematics." Actually they are needed to "complete" the complex number system and should be part of standard algebra and calculus. They should also be part of physics!

They have been used, indirectly, to define differentiation

$$df \equiv dx \, \frac{df(x)}{dx} \equiv (dx) \lim_{dx \to 0} \frac{f(x + dx) - f(x)}{dx} \equiv f(x + \delta) - f(x) \quad (7.1)$$

Here $f(x) \equiv y$ is a real number. Suppose we choose the function $f(x) \equiv x$, then $f(x + \delta) - f(x) = x + \delta - x = \delta = df = dy$. Thus $\{y\}$ should include the $\{\delta\text{'s}\}$, since $f(x) \equiv x$, evaluated at $x = (0 + \delta)$, would give $y = f(x) = 0 + \delta = \delta$.

Now suppose that we define a function $\psi(x)$, for which the values of ψ are *restricted* to the incredibles, $\{\delta, 0\delta, -\delta, i\delta, 0i\delta, -i\delta\}$ and x ranges over the reals *and* incredibles. Then $\psi(x_1) \neq \psi(x_2)$ in general for $x_1 \neq x_2$. However, $\psi(x_1)\psi(x_1) = (0i)$. In fact, $\psi(x_1)\psi(x_2) = (0i)$ or (0) for any x_1 and x_2. (!)

The incredibles, as defined so far, can be used to build a rigorous base for differential and integral calculus. This is valuable for physics, but *we need more.* The Pauli exclusion principle is responsible for the existence of chemistry (and even humans who can study the nature of existence). To include it in the physics, we apparently need special numbers: $\psi(x_1)\psi(x_2) \equiv -\psi(x_2)\psi(x_1) \neq 0$. We can try to define them as follows: Replace δ by δ_1. Then $(\delta_1)(\delta_1) \equiv (0i)$.

We have *assumed* that $i(\delta) = (i\delta) = (\delta)i$ in defining δ, but since all products involving δ give zero, we have had no real tests of this property. We now introduce $\delta_2 \neq \delta_1$, with $\delta_2\delta_2 \equiv (0i)$. Let us *assume* that $\delta_1\delta_2 \equiv I_{12} \equiv -\delta_2\delta_1 = -I_{12}^* = -I_{21} \neq (0i)$. Now $\delta_1 + \delta_1 \neq \delta_1$ and we generate an infinite set of δ_1's and δ_2's by *defining* $\delta_1\delta_2 \neq 0i$. Even $0i\delta_1$ might be distinct from $0i\delta_2$. We shall assume that $0i\delta_1 \equiv 0i\delta_2$. Our system now contains $R\delta_1, R\delta_2, -R\delta_1, -R\delta_2, R\delta i_1, Ri\delta_2, -Ri\delta_1, -Ri\delta_2, 0\delta_1 \equiv 0\delta_2, 0i\delta_1 \equiv 0i\delta_2$, where R is any integer, and perhaps we could extend this to the rationals.

Right away we see that *associativity is lost*, since $(\delta_1\delta_1)\delta_2 = (0i)\delta_2 \equiv (0i\delta_2)$, whereas $\delta_1(\delta_1\delta_2) = \delta_1 I_{12} = \delta_1(iR_{12}) = (i\delta_1)R_{12}$ and $R_{12} \neq 0$, by assumption. (Note that the 2 x 2 raising and lowering matrices are like this: $(\sigma_+\sigma_+)\sigma_- = (0)\sigma_- = 0$ but $\sigma_+(\sigma_+\sigma_-) = \sigma_+(1) = \sigma_+ \neq 0$. Matrix multiplication is not associative in this circumstance.) Loss of associativity is very serious. Schwinger keeps *both* associativity and distributivity. Without the distributive law, we cannot evaluate

$$(\delta_1 + \delta_2)(\delta_1 + \delta_2) = \delta_1\delta_1 + \delta_2\delta_1 + \delta_1\delta_2 + \delta_2\delta_2 = 0i + -\delta_1\delta_2 + \delta_1\delta_2 + 0i = 0i$$

$$\delta_1(\delta_2 + \delta_2) = \delta_1\delta_2 + \delta_1\delta_2 = I_{12} + I_{12} = 2I_{12} \neq I_{12} \neq 0i \tag{7.2}$$

so we must accept it as valid.

If $\delta_1\delta_2 \equiv \epsilon i$, then $(\delta_1\delta_2)(\delta_1\delta_2)(\delta_1\delta_2)(\delta_1\delta_2) = \epsilon^4$ and further products move toward 0 for $\epsilon < 1$, remain at 1 for $\epsilon = 1$, or move toward ∞ if $\epsilon > 1$. It is not clear what value of ϵ should be chosen. Unless $\epsilon \neq 0$, we have $\int \psi^\dagger \eta(dx) = \int 0(dx) = 0$, since $\psi(x)^a\eta(x')^c \equiv -\eta(x')^c\psi(x)^a$ in Schwinger's approach. It is also not clear at this point if there is a need for other numbers $\delta_3 \neq \delta_2 \neq \delta_1$, $\delta_3\delta_2 \equiv I_{32}$, etc., but see below.

Schwinger takes a more naive approach to the incredible numbers and never tries to actually "construct" one. He simply *assumes* $\eta(x)\eta(x') = -\eta(x')\eta(x)$. He assumes $[\eta(x)\eta(x')]^* = \eta(x')^*\eta(x)^*$. He assumes $[\eta_1(x) + \eta_2(x)]G[\eta_1(x') + \eta_2(x')] = \eta_1(x)G\eta_1(x') + \eta_2(x)G\eta_1(x') + \eta_1(x)G\eta_2(x') + \eta_2(x')G\eta_2(x')$. He assumes

$$\eta(p) \equiv \int (dx) \exp(-ip^\mu x_\mu)\eta(x) \tag{7.3}$$

is well defined and $\exp(-ip^\mu x_\mu)\eta(x) = \eta(x)\exp(-ip^\mu x_\mu)$. He assumes that in

$$\eta_{p\sigma} \equiv (2md\omega_p)^{1/2}u^*_{p\sigma}\eta(p) \tag{7.4}$$

the real number

$$(2md\omega_p)^{1/2} \equiv \left(2m\,\frac{d^3p}{(2\pi)^3}\,\frac{1}{2p^0}\right)^{1/2} \tag{7.5}$$

is meaningful here in determining the "value" of the complex incredible $\eta_{p\sigma}$, which is equivalent to assuming that $5\eta(p) \neq \eta(p)$. [Actually $u^*_{p\sigma}$ and $\eta(p)$ are 4 x 1 and 1 x 4 matrices; but this is not important here. We are interested in how the numbers in the matrices multiply. The incredible numbers $\eta_{p\sigma}$ are not matrices.] He obtains the physical amplitude for spin (σ) and momentum (p) eigen production and annihilation, under the action of source/sinks η_1 and η_2 in the form

$$\langle 0_+|0_-\rangle^{\eta_1+\eta_2} = \exp[iW(\eta)], \qquad W(\eta) \equiv \tfrac{1}{2}\int (dx)(dx')[\eta_1(x) + \eta_2(x)]G(x,x')$$
$$\times [\eta_1(x') + \eta_2(x')] \equiv \tfrac{1}{2}\int (dx)(dx')[\eta_1(x)G\eta_1(x') + \eta_1(x)G\eta_2(x')$$
$$+ \eta_2(x)G\eta_1(x') + \eta_2(x)G\eta_2(x')] \tag{7.6}$$

He then assumes for "physical" reasons that $\eta_1(x)G\eta_2(x') = \eta_2(x)G\eta_1(x')$, giving

$$\langle 0_+|0_-\rangle^{\eta_1+\eta_2} = \exp \frac{i}{2}\int (dx)(dx')\eta_1(x)G\eta_1(x') \exp i\int (dx)(dx')\eta_1(x)G\eta_2(x') \times \exp \frac{i}{2}\int (dx)(dx')\eta_2(x)G\eta_2(x') \tag{7.7}$$

$$\langle 0_+|0_-\rangle^{\eta_1+\eta_2} \equiv \langle 0_+|0_-\rangle^{\eta_1}\left[\exp i\int (dx)(dx')\eta_1(x)G\eta_2(x')\right]\langle 0_+|0_-\rangle^{\eta_2}$$

$$= \langle 0_+|0_-\rangle^{\eta_1}\left(\exp \sum_{p,\sigma} i\overset{*}{\eta}_{1p\sigma} i\eta_{2p\sigma}\right)\langle 0_+|0_-\rangle^{\eta_2}$$

He assumes $\eta_{p\sigma}\eta_{p'\sigma'} = -\eta_{p'\sigma'}\eta_{p\sigma}$, $\eta_{p\sigma}\overset{*}{\eta}_{p'\sigma'} = -\overset{*}{\eta}_{p'\sigma'}\eta_{p\sigma}$, and $\overset{*}{\eta}_{p\sigma}\overset{*}{\eta}_{p'\sigma'} = -\overset{*}{\eta}_{p'\sigma'}\overset{*}{\eta}_{p\sigma}$. [This probably can be deduced from equations (7.3) and (7.4).] Obviously, $\overset{*}{\eta}_{1p\sigma}\eta_{2p\sigma} \neq 0$, otherwise there is no interesting physics here. He then defines

$$\exp\left[\sum_{p,\sigma}(i\overset{*}{\eta}_{1p\sigma} i\eta_{2p\sigma})\right] \equiv \prod_{p\sigma} \exp\,(i\overset{*}{\eta}_{1p\sigma} i\eta_{2p\sigma}) \tag{7.8}$$

which is all right, since $(\overset{*}{\eta}_{1p\sigma}\eta_{2p\sigma})$ is an "ordinary" commuting complex number. Now he defines

$$\exp\,(i\overset{*}{\eta}_{1p\sigma} i\eta_{2p\sigma}) \equiv \sum_n^{\infty} \frac{1}{n!}(i\overset{*}{\eta}_{1p\sigma} i\eta_{2p\sigma})^n \tag{7.9}$$

The Fermi-Dirac statistics are obtained because $n = 0$ or 1 only. But this only follows by *assuming* associative multiplication

$$(i\overset{*}{\eta}_{1p\sigma} i\eta_{2p\sigma})(i\overset{*}{\eta}_{1p\sigma} i\eta_{2p\sigma}) \equiv i\overset{*}{\eta}_{1p\sigma}(i\eta_{2p\sigma} i\overset{*}{\eta}_{1p\sigma})i\eta_{2p\sigma} \equiv -i\overset{*}{\eta}_{1p\sigma}(i\overset{*}{\eta}_{1p\sigma} i\eta_{2p\sigma})i\eta_{2p\sigma}$$

$$\equiv -(i\overset{*}{\eta}_{1p\sigma} i\overset{*}{\eta}_{1p\sigma})(i\eta_{2p\sigma} i\eta_{2p\sigma}) = -(0i+0)(0i+0) = -(0+0i) \tag{7.10}$$

We see that he has assumed associative multiplication and $\overset{*}{\eta}_{1p\sigma}\eta_{2p\sigma} \neq 0$. (However, if this number is an ordinary complex number, then its square should not be zero!) We now have

$$\exp \sum_{p,\sigma}(i\overset{*}{\eta}_{1p\sigma} i\eta_{2p\sigma}) = \prod_{p,\sigma}(i\overset{*}{\eta}_{1p\sigma} i\eta_{2p\sigma})^{n_{p\sigma}}, \qquad n_{p\sigma} = 0 \text{ or } 1 \text{ only} \tag{7.11}$$

since $0! \equiv 1$ and $1! = 1$. Note again that $(\overset{*}{\eta}_{1p\sigma}\eta_{2p\sigma})$ is complex, commuting and its square is zero. He now assumes that the physical process of creation and annihilation, by the sources η_1 and η_2, can be expressed as

$$\langle 0_+|0_-\rangle^{\eta_1+\eta_2} \equiv \sum_{\{n\}} \langle 0_+|\{n\}\rangle^{\eta_1}\,\langle\{n\}|0_-\rangle^{\eta_2} \tag{7.12}$$

Then

$$\prod_a (i\overset{*}{\eta}_{1a} i\eta_{2a})^{n_a} = (i\overset{*}{\eta}_{1a} i\eta_{2a})^{n_a}(i\overset{*}{\eta}_{1b} i\eta_{2b})^{n_b}\cdots \tag{7.13}$$

Again he assumes the use of associative multiplication by writing

$$(i\overset{*}{\eta}_{1a} i\eta_{2a})(i\overset{*}{\eta}_{1b} i\eta_{2b}) \equiv i\overset{*}{\eta}_{1b}(i\overset{*}{\eta}_{1a} i\eta_{2a})i\eta_{2b} \equiv (i\overset{*}{\eta}_{1b} i\overset{*}{\eta}_{1a})(i\eta_{2a} i\eta_{2b}) \tag{7.14}$$

He also assumes $(\eta_1\eta_2)^0 = 1$, which is all right since $\eta_1\eta_2$ is an "ordinary" number. We, therefore, consider only occupied $p\sigma$ states in Π_a, which gives

$$\prod_a (i\eta_{1a}^* i\eta_{2a})^{n_a} = (\cdots i\eta_{1c}^* i\eta_{1b}^* i\eta_{1a}^*)(i\eta_{2a} i\eta_{2b} i\eta_{2c} \cdots) \tag{7.15}$$

Comparing equations (7.15), (7.12), (7.11), and (7.7), he now *assumes* the division

$$\langle \{a, b, c, \ldots\} | 0_- \rangle^{\eta_2} \equiv \langle 0_+ | 0_- \rangle^{\eta_2} (i\eta_{2a} i\eta_{2b} i\eta_{2c} \cdots)$$
$$\langle 0_+ | \{a, b, c, \ldots\} \rangle^{\eta_1} \equiv \langle 0_+ | 0_- \rangle^{\eta_1} (\cdots i\eta_{1c}^* i\eta_{1b}^* i\eta_{1a}^*) \tag{7.16}$$

is physical. Notice that $\langle 0_+ | 0_- \rangle$ involved only pairs of $\eta_1^*\eta_2$. It, therefore, is an "ordinary" complex number. But $\langle \{a, b, c\} | 0_- \rangle$ is proportional to the product of three η's. It therefore is an incredible number also! Schwinger defines

$$\langle \{a, b, c\} | 0_- \rangle^* \equiv \langle 0_- | \{a, b, c\} \rangle \tag{7.17}$$

in analogy with the usual quantum assumption. The absolute value

$$\langle \{a\} | 0_- \rangle \langle \{a\} | 0_- \rangle^* = |\langle 0_+ | 0_- \rangle|^2 i\eta_a(-i\eta_a^*) \tag{7.18}$$

and

$$i\eta_a(-i\eta_a^*) = -ii\eta_a\eta_a^* = (\eta_{aR} + i\eta_{aI})(\eta_{aR} - i\eta_{aI}) = 0 + i\eta_{aI}\eta_{aR} - i\eta_{aR}\eta_{aI} + 0$$
$$= 2i\eta_{aI}\eta_{aR} = 2(i\eta_{aI}\eta_{aR}) = \text{"ordinary" real number} \tag{7.19}$$

since an imaginary incredible times a real incredible, gives a real number. We see that the vector space $|\rangle$ must be generalized to include pseudoincredible vector coefficients.

Notice in equation (7.16) that $\eta_a \neq \eta_b \neq \eta_c$ because the states created (or destroyed), a, b, c, are distinct states. Also we do not want the product $\eta_a\eta_b\eta_c \cdots$ to be zero. We should expect that the probability amplitude for creating a large number of quanta is smaller than that for creating a few quanta. This suggests that $\eta_a\eta_a^* < 1$. This is also required by Schwinger's assumption

$$\prod_{\{n\}} \langle 0_- | \{n\} \rangle^{\eta} \langle \{n\} | 0_- \rangle^{\eta} \equiv \langle 0_- | 0_- \rangle \equiv 1 = \langle 0_+ | 0_+ \rangle^{\eta} \tag{7.20}$$

Since $\eta_a\eta_a^* = 2i\eta_{aI}\eta_{aR}$, we conclude that $(i\eta_{aI})\eta_{aR} < \frac{1}{2}$. Since $\{n\}$ is a sum over an infinite number of momentum states, we see that actually $(i\eta_{aI})\eta_{aR} \approx 0$. Therefore, this product may be something as pathological as Dirac's delta function $\delta(x) = 0$ for $x \neq 0$ yet $\int \delta(x)dx = 1$. It appears then that we should define an infinite set of pseudoincredibles $\delta_1, \delta_2, \delta_3, \ldots$, with $\delta_m\delta_n \neq 0i$ unless $m = n$, but $\delta_m\delta_n \approx 0i$ and $(0i)\delta_m \neq 0$, whatever that means! Schwinger's numbers are *not* incredibles. Both the anticommuting η_a and commuting $(\eta_a\eta_b)$ numbers can have zero for their square. They are completely distinct from complex numbers. It also appears that all the computational content of quantum electrodynamics can be extracted without ever facing up to the "numerical value" of $\eta\eta^*$, since Schwinger claims his anticommuting number approach gives *all* the results obtained by the older operator field approach.

The next step is to relate, mathematically, the sources to particular physical situations. These "situations" take place in 3-space but create 4-space propagating quanta. The problem is to convert this into a numerically productive idea, whereby the masses of the detected quanta can be predicted.

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References

Edmonds, J. D., Jr., (1974). *International Journal of Theoretical Physics*, **10**, 115; **10**, 273; **11**, 309; *American Journal of Physics*, **42**, 220; *Foundation of Physics*, **4**, 473.

Edmonds, J. R., Jr. (1975). *International Journal of Theoretical Physics*, **13**, 259; **13**, 297; **13**, 431. *Foundations of Physics*, **5**, 239. *Lettere al Nuovo Cimento*, **13**, 185.

Schwinger, J. (1970). *Particles, Sources and Fields.* (Addison-Wesley, Reading, Massachusetts).